

Incidences between points and non-coplanar circles*

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Abstract

We establish an improved upper bound for the number of incidences between m points and n arbitrary circles in three dimensions. The previous best known bound, which applies in any dimension, is $O^*(m^{2/3}n^{2/3} + m^{6/11}n^{9/11} + m + n)$. Since all the points and circles may lie on a common plane (or sphere), it is impossible to improve the three-dimensional bound without improving the two-dimensional one.

Nevertheless, we show that if the set of circles is required to be “truly three-dimensional” in the sense that there exists a $q < n$ so that no sphere or plane contains more than q of the circles, then the bound can be improved to

$$O^*(m^{3/7}n^{6/7} + m^{2/3}n^{1/2}q^{1/6} + m^{6/11}n^{15/22}q^{3/22} + m + n).$$

For various ranges of parameters (e.g., when $m = \Theta(n)$ and $q = o(n^{7/9})$), this bound is smaller than the best known two-dimensional lower bound $\Omega^*(m^{2/3}n^{2/3} + m + n)$. Thus we obtain an incidence theorem analogous to the one in the recent distinct distances paper by Guth and Katz, which states that if we have a collection of points and lines in \mathbb{R}^3 and we restrict the number of lines that can lie on a common plane or regulus, then the maximum number of point-line incidences is smaller than the maximum number of incidences that can occur in the plane.

Our result is obtained by applying the polynomial partitioning technique of Guth and Katz using a constant-degree partitioning polynomial, as was also recently used by Solymosi and Tao. We also rely on various additional tools from analytic, algebraic, and combinatorial geometry.

1 Introduction

Recently, Guth and Katz [16] presented the *polynomial partitioning technique* as a major technical tool in their solution of the famous planar distinct distances problem of Erdős [14]. This problem can be reduced to an incidence problem involving points and lines in \mathbb{R}^3 (following the reduction that was proposed in [13]), which can be solved by applying the

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aforementioned polynomial partitioning technique. The Guth-Katz result prompted various other incidence-related studies that rely on polynomial partitioning (e.g., see [18, 19, 27, 31]). One consequence of these studies is that they have led to further developments and enhancements of the technique itself (as seen for example in the interesting use of induction in [27], and the use of two partitioning polynomials in [18, 31]). Also, the technique was recently applied to some problems that are not incidence related: it was used to reprove the existence of spanning trees with small crossing number in any dimension [19], and to obtain improved algorithms for range searching with semialgebraic sets [2]. Thus, it seems fair to say that applications and enhancements of the polynomial partitioning technique are an active area of research in combinatorial geometry.

In this paper we study incidences between points and circles in three dimensions. Let \mathcal{P} be a set of m points and \mathcal{C} a set of n circles in \mathbb{R}^3 . We denote the number of point-circle incidences in $\mathcal{P} \times \mathcal{C}$ as $I(\mathcal{P}, \mathcal{C})$. When the circles have arbitrary radii, the current best bound for any dimension $d \geq 2$ (due to Aronov et al. [7]) is¹

$$I(\mathcal{P}, \mathcal{C}) = O^* \left(m^{2/3} n^{2/3} + m^{6/11} n^{9/11} + m + n \right). \quad (1)$$

Here the $O^*(\cdot)$ notation hides polylogarithmic factors; the precise best known upper bound is $O(m^{2/3} n^{2/3} + m^{6/11} n^{9/11} \log^{2/11}(m^3/n) + m + n)$ [21].

Since the three-dimensional case also allows \mathcal{P} and \mathcal{C} to lie on a single common plane (or sphere²), the point-circle incidence bound in \mathbb{R}^3 cannot be improved without first improving the planar bound (which is an open problem for almost 10 years). Nevertheless, as we show in this paper, an improved bound can be obtained if the configuration of points and circles is “truly three-dimensional” in the sense that no sphere or plane contains too many circles from \mathcal{C} (Guth and Katz [16] use a similar assumption on the maximum number of lines that can lie in a common plane or regulus). Our main result is the following:

Theorem 1.1. *Let \mathcal{P} be a set of m points and let \mathcal{C} be a set of n circles in \mathbb{R}^3 , let ε be an arbitrarily small positive constant, and let $q < n$ be an integer. If no sphere or plane contains more than q circles of \mathcal{C} , then*

$$I(\mathcal{P}, \mathcal{C}) = O \left(m^{3/7+\varepsilon} n^{6/7} + m^{2/3+\varepsilon} n^{1/2} q^{1/6} + m^{6/11+\varepsilon} n^{15/22} q^{3/22} + m + n \right), \quad (2)$$

where the constant of proportionality depends on ε .

Remarks. (1) In the planar case, the best known lower bound for the number of point-circle incidences is $\Omega^*(m^{2/3} n^{2/3} + m + n)$ (e.g., see [25])³. Theorem 1.1 implies that for certain ranges of m, n , and q , a better upper bound holds in \mathbb{R}^3 . This is the case, for example, when $m = \Theta(n)$ and $q = o(n^{7/9})$.

(2) When $m > n^{3/2}$, we have $m^{3/7} n^{6/7} < m$ and $m^{6/11} n^{15/22} q^{3/22} < m^{2/3} n^{1/2} q^{1/6}$. Hence, we have

$$I(\mathcal{P}, \mathcal{C}) = O(m^{2/3+\varepsilon} n^{1/2} q^{1/6} + m^{1+\varepsilon}).$$

Moreover, for $q = O(m^2/n^3)$ we have $I(\mathcal{P}, \mathcal{C}) = O(m^{1+\varepsilon})$.

¹In the notation $O^*(\cdot)$, we neglect subpolynomial factors of the form $O(m^\varepsilon)$, for any $\varepsilon > 0$.

²There is no real difference between the cases of coplanarity and cosphericity of the points and circles, since the latter case can be reduced to the former (and vice versa) by means of the stereographic projection.

³It is in fact larger than the stated expression by a fractional logarithmic factor.

(3) When $m \leq n^{3/2}$, any of the terms except for m can dominate the bound. However, if in addition $q = O\left(\left(\frac{n^3}{m^2}\right)^{3/7}\right)$ then the bound becomes

$$I(\mathcal{P}, \mathcal{C}) = O(m^{3/7+\varepsilon} n^{6/7} + n).$$

Note also that the interesting range of parameters is $m = \Omega^*(n^{1/3})$ and $m = O^*(n^2)$; in the complementary ranges both the old and new bounds become linear in $m + n$. In the interesting range, the new bound is asymptotically smaller than the planar bound given in (1) for q sufficiently small, as specified above.

The proof of Theorem 1.1 is based on the polynomial partitioning technique of Guth and Katz [16], where we use a constant-degree partitioning polynomial in a manner similar to that used by Solymosi and Tao [27] (the use of constant-degree polynomials and the induction arguments it leads to are essentially the only similarities with the technique of [27], which does not apply to circles in any dimension, since it cannot handle situations where arbitrarily many curves can pass between any specific pair of points). The application of this technique to incidences involving circles leads to new problems involving the handling of points that are incident to many circles that are entirely contained in the zero set of the partitioning polynomial. To handle this situation we turn these circles to lines using an inversion transformation. We then analyze the geometric and algebraic structure of the transformed zero set using a variety of tools such as *flecnode polynomials* (as used in [16]), classical 19th-century results in analytic geometry from [26], a very recent result about surfaces that are “ruled” by lines and circles [24], and some tools from traditional combinatorial geometry.

An application: similar triangles. Given a set \mathcal{P} of points in \mathbb{R}^3 and a triangle Δ , we denote by $F(\mathcal{P}, \Delta)$ the number of triangles that are spanned by points of \mathcal{P} and similar to Δ . Let $F(m) = \max_{|\mathcal{P}|=m, \Delta} F(\mathcal{P}, \Delta)$. The problem of obtaining good bounds for $F(m)$ is motivated by questions in exact pattern matching (see e.g. [1, 5, 6, 9]). Theorem 1.1 implies the bound $F(m) = O^*(m^{15/7})$, which slightly improves upon the previous bound of $O^*(m^{58/27})$ from [6] (in the older bound, the $O^*(\cdot)$ notation only hides polylogarithmic factors). The new bound is an almost immediate corollary of Theorem 1.1, while the previous bound requires a more complicated analysis. This is discussed further in Section 5.

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2 Algebraic preliminaries

We will briefly review some results from algebraic geometry.

Polynomial partitioning. Consider a set \mathcal{P} of m points in \mathbb{R}^d . Given a polynomial $f \in \mathbb{R}[x_1, \dots, x_d]$, we define the *zero set* of f to be $Z(f) = \{p \in \mathbb{R}^d \mid f(p) = 0\}$. For $1 < r \leq m$, we say that $f \in \mathbb{R}[x_1, \dots, x_d]$ is an *r -partitioning polynomial* for \mathcal{P} if no connected component of $\mathbb{R}^d \setminus Z(f)$ contains more than m/r points of \mathcal{P} . Notice that there is no restriction on the number of points of \mathcal{P} that lie in $Z(f)$.

The following result is due to Guth and Katz [16]. A detailed proof can also be found in [19].

Theorem 2.1. (Polynomial partitioning [16]) *Let \mathcal{P} be a set of m points in \mathbb{R}^d . Then for every $1 < r \leq m$, there exists an r -partitioning polynomial $f \in \mathbb{R}[x_1, \dots, x_d]$ of degree $O(r^{1/d})$.*

This theorem asserts the existence of a polynomial of reasonably small degree such that its zero set partitions the space into maximal connected cells (connected components of $\mathbb{R}^d \setminus Z(f)$), each containing roughly the same number of points. To use such a partitioning, we also need a bound on the maximum possible number of cells. Such a bound is provided by the following theorem, which is a refined variant of the Milnor-Thom theorem [23, 28].

Theorem 2.2. (Warren's theorem [29]) *Let f_1, \dots, f_n be polynomials in $\mathbb{R}[x_1, \dots, x_d]$, each of degree at most k , and put $V = \mathbb{R}^d \setminus \bigcup_{i=1}^n Z(f_i)$. Then the number of connected components of V is at most $2(2k)^d \sum_{i=0}^d \binom{n}{i} 2^i$.*

Consider an r -partitioning polynomial f for a point-set \mathcal{P} , as provided in Theorem 2.1. The number of cells in the partition is equal to the number of connected components of $\mathbb{R}^d \setminus Z(f)$. By Theorem 2.2, this is $O((r^{1/d})^d) = O(r)$ (recall that f is of degree $O(r^{1/d})$).

Since this paper studies incidences in a three-dimensional space, we will only apply the above theorems for the case $d = 3$.

We also need the following basic property of zero sets of polynomials in the plane (for further discussion see [10, 11]).

Theorem 2.3. (Bézout's theorem) *Let f, g be two bivariate polynomials in $\mathbb{R}[x_1, x_2]$ of degrees D_f and D_g , respectively. (i) If $Z(f)$ and $Z(g)$ have a finite number of common points, then this number is at most $D_f \cdot D_g$. (ii) If $Z(f)$ and $Z(g)$ have an infinite number of common points, then f and g have a common factor.*

The following lemma is proved in [12, Proposition 1] and [15, Corollary 2.5].

Lemma 2.4. (Guth and Katz [15]) *Let f and g be two trivariate polynomials of respective degrees D_f and D_g , such that f and g have no common factor. Then there are at most $D_f \cdot D_g$ lines on which both f and g vanish identically.*

Flecnode polynomial. Given a polynomial $f \in \mathbb{R}[x_1, x_2, x_3]$ of degree $d \geq 3$, a *flecnode* is a point $p \in Z(f)$ for which there exists a line that passes through p and agrees with $Z(f)$ at p to order three. That is, if the direction of the line is v then

$$f(p) = 0, \quad \nabla_v f(p) = 0, \quad \nabla_v^2 f(p) = 0, \quad \nabla_v^3 f(p) = 0,$$

where $\nabla_v f, \nabla_v^2 f, \nabla_v^3 f$ are, respectively, the first, second, and third-order derivatives of f in the direction v . That is $\nabla_v f = \nabla f \cdot v$, $\nabla_v^2 f = v^T H_f v$, where H_f is the *Hessian matrix* of f , and $\nabla_v^3 f$ is similarly defined, although its explicit expression is somewhat more involved.

The *flecnode polynomial* of f , denoted FL_f , is the polynomial obtained by eliminating v from these four equations. Note that the last three polynomials of the system are homogeneous in v . We thus have a system of four equations in six variables, which is homogeneous in the last three. Eliminating those variables results in a single polynomial equation in $p = (x_1, x_2, x_3)$. The flecnode polynomial of f vanishes on all the flecnodes of f . The following results are mentioned in [16, Section 3] and are taken from [26, Chapter XVII, Section III].

Lemma 2.5. *Given a polynomial $f \in \mathbb{R}[x_1, x_2, x_3]$ of degree $d \geq 3$, its flecnode polynomial FL_f has degree at most $11d - 24$.*

Definition 2.6. An algebraic surface $S \subset \mathbb{R}^3$ is said to be *ruled* if every point of S is incident to a straight line that is fully contained in S . Equivalently, S is a union of lines.⁴ We say an irreducible surface is *triply ruled* if for every point on the surface there are three straight lines contained in the surface that pass through that point. The only triply ruled surfaces are planes. We say an irreducible surface is *doubly ruled* if it is not triply ruled and for every point on the surface there are two straight lines contained in the surface that pass through that point. The only doubly ruled surfaces are the hyperbolic paraboloid and the single-sheeted hyperboloid (e.g., see [16]). Finally, we say an irreducible ruled surface is *singly ruled* if it is not doubly or triply ruled.

Lemma 2.7. *Given a polynomial $f \in \mathbb{R}[x_1, x_2, x_3]$ of degree $d \geq 3$, every line that is fully contained in $Z(f)$ is also fully contained in $Z(\text{FL}_f)$.*

Proof. Every point on any such line is a flecnode of f , so FL_f vanishes identically on the line. □

Theorem 2.8. (Cayley-Salmon [26]) *Let $f \in \mathbb{R}[x_1, x_2, x_3]$ be a polynomial of degree $d \geq 3$. Then $Z(f)$ is ruled if and only if $Z(f) \subseteq Z(\text{FL}_f)$.*

Corollary 2.9. *Let $f \in \mathbb{R}[x_1, x_2, x_3]$ be an irreducible polynomial of degree $d \geq 3$. If $Z(f)$ contains more than $d(11d - 24)$ lines then $Z(f)$ is a ruled surface.*

Proof. Lemma 2.4 and Lemma 2.7 imply that f and FL_f have a common factor. Since f is irreducible, f divides FL_f , and Theorem 2.8 completes the proof. □

3 The main theorem

In this section we prove Theorem 1.1, which we restate for the convenience of the the reader.

Theorem 1.1 *Let \mathcal{P} be a set of m points and let \mathcal{C} be a set of n circles in \mathbb{R}^3 , let ε be an arbitrarily small positive constant, and let $q < n$ be an integer. If no sphere or plane contains more than q circles of \mathcal{C} , then*

$$I(\mathcal{P}, \mathcal{C}) = O\left(m^{3/7+\varepsilon}n^{6/7} + m^{2/3+\varepsilon}n^{1/2}q^{1/6} + m^{6/11+\varepsilon}n^{15/22}q^{3/22} + m + n\right),$$

⁴We do not insist on the more restrictive definition used in differential geometry, which requires the ruling lines to form a smooth 1-parameter family; cf. [8, Chapter III].

where the constant of proportionality depends on ε .

Proof. We prove the theorem by induction on $m + n$. Specifically, we prove by induction that for any fixed $\varepsilon > 0$,

$$I(\mathcal{P}, \mathcal{C}) \leq \alpha_1 \left(m^{3/7+\varepsilon} n^{6/7} + m^{2/3+\varepsilon} n^{1/2} q^{1/6} + m^{6/11+\varepsilon} n^{15/22} q^{3/22} \right) + \alpha_2(m + n)$$

for sufficiently large constants α_1, α_2 that depend on ε .

We start by recalling a well-known simple, albeit weaker bound. Let $G = (V_1, V_2, E)$ be the bipartite graph such that V_1 contains a vertex for every point in \mathcal{P} , V_2 contains a vertex for every circle in \mathcal{C} , and there is an edge $e = (u, v) \in E$ if the point that corresponds to u is incident to the circle that corresponds to v . Since two circles have at most two intersection points, G cannot contain $K_{3,2}$ as a subgraph. Applying the Kővári-Sós-Turán theorem (e.g., see [22, Section 4.5]) implies $I(\mathcal{P}, \mathcal{C}) = O(n^{2/3}m + n)$. This immediately implies the theorem if $m = O(n^{1/3})$. Thus we may assume that $n = O(m^3)$.

We next apply the polynomial partitioning technique. Specifically, we set r as a sufficiently large constant (whose value depends on ε and will be determined later), and apply the polynomial partitioning theorem (Theorem 2.1) to obtain an r -partitioning polynomial f . According to the theorem, f is of degree $D = O(r^{1/3})$ and $Z(f)$ partitions \mathbb{R}^3 into maximal connected cells, each containing at most m/r points of \mathcal{P} . As already noted, Warren's theorem (Theorem 2.2) implies that the number of cells is $O(r)$.

Let \mathcal{C}_0 denote the subset of circles of \mathcal{C} that are fully contained in $Z(f)$, and let $\mathcal{C}' = \mathcal{C} \setminus \mathcal{C}_0$. Similarly, set $\mathcal{P}_0 = \mathcal{P} \cap Z(f)$ and $\mathcal{P}' = \mathcal{P} \setminus \mathcal{P}_0$. Notice that

$$I(\mathcal{P}, \mathcal{C}) = I(\mathcal{P}_0, \mathcal{C}_0) + I(\mathcal{P}_0, \mathcal{C}') + I(\mathcal{P}', \mathcal{C}'). \quad (3)$$

The terms $I(\mathcal{P}_0, \mathcal{C}')$ and $I(\mathcal{P}', \mathcal{C}')$ can be bounded using techniques that are by now fairly standard. On the other hand, bounding $I(\mathcal{P}_0, \mathcal{C}_0)$ is the main technical challenge in this proof. Other works that have applied the polynomial partitioning technique, such as [18, 19, 27, 31, 32], also spend most of their efforts on curves that are fully contained in the zero set of the partitioning polynomial (where these curves are either original input curves or the intersection of input surfaces with the zero set).

Bounding $I(\mathcal{P}_0, \mathcal{C}')$ and $I(\mathcal{P}', \mathcal{C}')$. For a circle $C \in \mathcal{C}'$, let Π_C be the plane that contains C , and let f_C denote the restriction of f to Π_C . Since C is not contained in $Z(f_C)$, f_C and the irreducible quadratic equation of C within Π_C do not have any common factor. Thus by Bézout's theorem (Theorem 2.3), C and $Z(f_C)$ have at most $2 \cdot \deg(f_C) = O(r^{1/3})$ common points. This immediately implies

$$I(\mathcal{P}_0, \mathcal{C}') = O(nr^{1/3}). \quad (4)$$

Next, let us denote the cells of the partition as K_1, \dots, K_s (recall that $s = O(r)$). For $i = 1, \dots, s$, put $\mathcal{P}_i = \mathcal{P} \cap K_i$ and let \mathcal{C}_i denote the set of the circles of \mathcal{C}' that intersect K_i . Put $m_i = |\mathcal{P}_i|$, $n_i = |\mathcal{C}_i|$, $m_{\text{cells}} = \sum_{i=1}^s m_i$, and recall that $m_i \leq m/r$ for every i . The above bound of $O(r^{1/3})$ on the number of intersection points of a circle $C \in \mathcal{C}'$ and $Z(f)$ implies that each circle crosses $O(r^{1/3})$ cells (a circle has to intersect $Z(f)$ when moving from one cell to another). This implies $\sum_i n_i = O(nr^{1/3})$.

Notice that $I(\mathcal{P}', \mathcal{C}') = \sum_{i=1}^s I(\mathcal{P}_i, \mathcal{C}_i)$, so we proceed to bound the number of incidences within a cell K_i . From the induction hypothesis, we get

$$\begin{aligned}
I(\mathcal{P}', \mathcal{C}') &\leq \sum_{i=1}^s \left(\alpha_1 \left(m_i^{3/7+\varepsilon} n_i^{6/7} + m_i^{2/3+\varepsilon} n_i^{1/2} q^{1/6} + m_i^{6/11+\varepsilon} n_i^{15/22} q^{3/22} \right) + \alpha_2 (m_i + n_i) \right) \\
&\leq \sum_{i=1}^s \left(\alpha_1 \left(\left(\frac{m}{r} \right)^{3/7+\varepsilon} n_i^{6/7} + \left(\frac{m}{r} \right)^{2/3+\varepsilon} n_i^{1/2} q^{1/6} + \left(\frac{m}{r} \right)^{6/11+\varepsilon} n_i^{15/22} q^{3/22} \right) \right) \\
&\quad + \alpha_2 \left(m_{\text{cells}} + \sum_{i=1}^s n_i \right).
\end{aligned} \tag{5}$$

Since $\sum_i n_i = O(nr^{1/3})$, Hölder's inequality implies

$$\begin{aligned}
\sum_{i=1}^s n_i^{6/7} &= O \left(\left(nr^{1/3} \right)^{6/7} \cdot r^{1/7} \right) = O \left(n^{6/7} r^{3/7} \right), \\
\sum_{i=1}^s n_i^{1/2} &= O \left(\left(nr^{1/3} \right)^{1/2} \cdot r^{1/2} \right) = O \left(n^{1/2} r^{2/3} \right), \\
\sum_{i=1}^s n_i^{15/22} &= O \left(\left(nr^{1/3} \right)^{15/22} \cdot r^{7/22} \right) = O \left(n^{15/22} r^{6/11} \right).
\end{aligned} \tag{6}$$

By combining (5) and (6), we obtain

$$\begin{aligned}
I(\mathcal{P}', \mathcal{C}') &\leq \alpha_1 \cdot O \left(\frac{m^{3/7+\varepsilon} n^{6/7} + m^{2/3+\varepsilon} n^{1/2} q^{1/6} + m^{6/11+\varepsilon} n^{15/22} q^{3/22}}{r^\varepsilon} \right) \\
&\quad + \alpha_2 \left(m_{\text{cells}} + O \left(nr^{1/3} \right) \right).
\end{aligned}$$

Notice that the bound in (4) is subsumed in this bound, and is dominated by $O(m^{3/7} n^{6/7})$ since we assume that $n = O(m^3)$. Taking r to be sufficiently large, and α_1 to be sufficiently larger than $\alpha_2 r^{1/3}$, we have, say,

$$I(\mathcal{P}_0 \cup \mathcal{P}', \mathcal{C}') \leq \frac{\alpha_1}{100} \left(m^{3/7+\varepsilon} n^{6/7} + m^{2/3+\varepsilon} n^{1/2} q^{1/6} + m^{6/11+\varepsilon} n^{15/22} q^{3/22} \right) + \alpha_2 m_{\text{cells}}. \tag{7}$$

Bounding $I(\mathcal{P}_0, \mathcal{C}_0)$: Handling shared points. We are left with the task of bounding the number of incidences between points of \mathcal{P} that are contained in $Z(f)$ and circles of \mathcal{C} that are fully contained in $Z(f)$. We call a point of \mathcal{P}_0 *shared* if it is contained in the zero sets of at least two distinct irreducible factors of f , and otherwise we call it *private*. We first consider the case of shared points.

Let \mathcal{P}_s denote the subset of points in \mathcal{P}_0 that are shared, and put $m_s = |\mathcal{P}_s|$. Let $f' = \nabla_e f$, where e is a generic choice of unit vector. Then $\deg(f') < D$ and $Z(f')$ contains the singular set of $Z(f)$. By definition, a shared point is a singular point of f , and thus $\mathcal{P}_s \subset Z(f) \cap Z(f')$. We may assume that f is square-free, which implies that $Z(f)$ and $Z(f')$ have no common factors. Note that at most $D^2/2$ circles can be contained in $Z(f) \cap Z(f')$. Indeed, a generic projection of $Z(f) \cap Z(f')$ onto \mathbb{R}^2 yields a (planar) algebraic curve of degree at most D^2 , and every circle contained in $Z(f) \cap Z(f')$ is a distinct ellipse contained in the projected curve. A planar algebraic curve of degree at most D^2 can contain at

most $D^2/2$ ellipses. Thus there are at most $\frac{D^2}{2}m_s$ incidences between points in \mathcal{P}_s and circles contained in $Z(f) \cap Z(f')$.

It remains to bound the number of incidences between points in \mathcal{P}_s and circles of \mathcal{C}_0 not contained in $Z(f) \cap Z(f')$. Consider such a circle C and let Π_C be the plane containing C . The intersection $Z(f') \cap \Pi_C$ is a planar algebraic curve of degree at most $D - 1$, and by assumption this curve does not contain C (since otherwise we would have $C \subset Z(f) \cap Z(f')$). According to Bézout's theorem (Theorem 2.3), C intersects $Z(f') \cap \Pi_C$ at most $2D - 2$ times, so C meets $Z(f) \cap Z(f')$ at most $2D - 2$ times. This in turn implies that $|C \cap \mathcal{P}_s| < 2D$. Therefore, by taking α_1 and α_2 to be sufficiently large, we have (recalling that we assume that $n = O(m^3)$)

$$I(\mathcal{P}_s, \mathcal{C}_0) \leq \frac{D^2}{2}m_s + n \cdot 2D^2 \leq \alpha_2 m_s + \frac{\alpha_1}{100} m^{3/7} n^{6/7}. \quad (8)$$

Bounding $I(\mathcal{P}_0, \mathcal{C}_0)$: Handling private points. Let $\mathcal{P}_p = \mathcal{P}_0 \setminus \mathcal{P}_s$ denote the set of private points in \mathcal{P}_0 . Recall that each private point is contained in the zero set of a single factor of f . Let f_1, f_2, \dots, f_t be the factors of f whose zero sets are planes or spheres. For $i = 1, \dots, t$, set $\mathcal{P}_{p,i}^{(1)} = \mathcal{P}_p \cap Z(f_i)$ and $m_{p,i} = |\mathcal{P}_{p,i}^{(1)}|$. Let $m_p^{(1)} = \sum_{i=1}^t m_{p,i}$ and let $n_{p,i}$ denote the number of circles of \mathcal{C}_0 that are fully contained in $Z(f_i)$. Put $\mathcal{P}_p^{(1)} = \bigcup_{i=1}^t \mathcal{P}_{p,i}^{(1)}$. Notice that $t = O(r^{1/3})$ and that $n_{p,i} \leq q$ for every i . Applying (1), we obtain

$$\begin{aligned} I(\mathcal{P}_p^{(1)}, \mathcal{C}_0) &= \sum_{i=1}^t O^* \left(m_{p,i}^{2/3} n_{p,i}^{2/3} + m_{p,i}^{6/11} n_{p,i}^{9/11} + m_{p,i} + n_{p,i} \right) \\ &= \sum_{i=1}^t O^* \left(m_{p,i}^{2/3} q^{2/3} + m_{p,i}^{6/11} q^{9/11} + m_{p,i} + q \right) \\ &= O^* \left(m^{2/3} q^{2/3} r^{1/9} + m^{6/11} q^{9/11} r^{5/33} + m_p^{(1)} + q r^{1/3} \right), \end{aligned}$$

where the last step uses Hölder's inequality. Since $q \leq n$ and there are no hidden polylogarithmic terms in the linear part of (1), it follows that when n, α_1 and α_2 are sufficiently large, we have

$$I(\mathcal{P}_p^{(1)}, \mathcal{C}_0) \leq \frac{\alpha_1}{100} \left(m^{2/3+\varepsilon} n^{1/2} q^{1/6} + m^{6/11+\varepsilon} n^{15/22} q^{3/22} \right) + \frac{\alpha_2}{2} (m_p + n). \quad (9)$$

Let $\mathcal{P}_p^{(2)} = \mathcal{P}_p \setminus \mathcal{P}_p^{(1)}$ be the set of private points that lie on the zero sets of factors of f which are neither planes nor spheres, and put $m_p^{(2)} = |\mathcal{P}_p^{(2)}|$. Let $g_1, \dots, g_{t'}$ be these factors of f , and let D_i denote the degree of g_i , $i = 1, \dots, t'$. To handle this case we require the following lemma, which constitutes a major component of our analysis and which is proved in Section 4. First, a definition.

Definition 3.1. Let $Z(g) \subset \mathbb{R}^3$ be an irreducible algebraic surface. We say that a point $p \in Z(g)$ is *popular* if it is incident to at least $44(\deg g)^2$ circles that are fully contained in $Z(g)$.

Lemma 3.2. *An irreducible algebraic surface that is neither a plane nor a sphere cannot contain more than two popular points.*

The lemma implies that the number of incidences between popular points of $\mathcal{P}_p^{(2)}$ and circles of \mathcal{C}_0 is at most $2(D/2)n = Dn \leq \alpha_2 n/2$ (the latter inequality holds if α_2 is chosen sufficiently large). The number of incidences between non-popular points of $\mathcal{P}_p^{(2)}$ and circles of \mathcal{C}_0 is at most $m_p^{(2)} \cdot 44D^2 \leq \alpha_2 m_p^{(2)}$ (again for a sufficiently large value of α_2). Combining this with (3), (7), (8), and (9), we get

$$I(\mathcal{P}, \mathcal{C}) \leq \alpha_1 \left(m^{3/7+\varepsilon} n^{6/7} + m^{2/3+\varepsilon} n^{1/2} q^{1/6} + m^{6/11+\varepsilon} n^{15/22} q^{3/22} \right) + \alpha_2(m + n).$$

This establishes the induction step, and thus completes the proof of the theorem. \square

4 The number of popular points in an irreducible surface

The proof of Lemma 3.2 uses the three-dimensional *inversion transformation* $I : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ about the origin (e.g., see [17, Chapter 37]). The transformation $I(\cdot)$ maps the point $p = (x_1, x_2, x_3) \neq (0, 0, 0)$ to the point $\bar{p} = I(p) = (\bar{x}_1, \bar{x}_2, \bar{x}_3)$, where

$$\bar{x}_i = \frac{x_i}{x_1^2 + x_2^2 + x_3^2}, \quad i = 1, 2, 3.$$

A proof for the following lemma can be found in [17, Chapter 37].

Lemma 4.1. (a) *Let C be a circle incident to the origin. Then $I(C)$ is a line.*
(b) *Let C be a circle not incident to the origin. Then $I(C)$ is a circle.*

Proof of Lemma 3.2. Consider an irreducible surface $Z = Z(g)$ which is neither a plane nor a sphere, and let $E = \deg(g)$. Assume, for contradiction, that there exist three popular points $z_1, z_2, z_3 \in Z$. By translating the axes we may assume that z_1 is the origin. We apply the inversion transformation to Z . Since I is its own inverse, $I(Z)$ can be written as $Z(g \circ I)$. To turn $g \circ I$ into a polynomial, we clear the denominators resulting from this transformation by multiplying $g \circ I$ by a suitable (minimal) power of $x_1^2 + x_2^2 + x_3^2$. This does not change the zero set of $g \circ I$ (except for possibly adding the origin 0 to the set). We refer to the resulting polynomial as \bar{g} . Notice that \bar{g} is of degree at most $2E$, since in order to clear denominators we must multiply $g \circ I$ by $(x_1^2 + x_2^2 + x_3^2)^E$, and the highest degree term will be the one that was previously the constant term (if it exists). We may assume that \bar{g} is not divisible by $x_1^2 + x_2^2 + x_3^2$ since eliminating such a factor will not affect $Z(\bar{g})$, except for possibly removing the origin from it. Indeed, if some other polynomial divided \bar{g} , then after applying the inversion again and clearing denominators we would obtain a polynomial that divides g . Thus, since g is irreducible, so is (the reduced) \bar{g} .

Since there are $44E^2$ circles incident to 0, Lemma 4.1(a) implies that $Z(\bar{g})$ contains at least $44E^2$ lines. We next claim that $Z(\bar{g})$ is a ruled surface. If $\deg(\bar{g}) \leq 2$, then we can consider all types of quadratic polynomials and observe that the ones whose zero sets may contain more than 44 lines are ruled. If $\deg(\bar{g}) \geq 3$, then since \bar{g} is irreducible of degree at most $2E$ and $Z(\bar{g})$ contains at least $44E^2 > 2E(11 \cdot 2E - 24)$ lines, Corollary 2.9 implies that $Z(\bar{g})$ is ruled.

Thus, Z can be written as the union of a set of circles and a (possibly empty) set of lines, all of which are incident to z_1 (observe that lines through z_1 are mapped to themselves by the inversion). By a symmetric argument, this property also holds for z_2 and z_3 . This implies that, for $i = 1, 2, 3$, every point u in Z is incident to a circle or a line that is also

incident to z_i . Recall that the only doubly ruled surfaces are the hyperbolic paraboloid and the hyperboloid of one sheet. Since both of these surfaces do not contain a point that is incident to infinitely many lines or circles contained in the surface, we conclude that Z is not doubly ruled.

We define a point $u \in Z$ to be *exceptional* if there are infinitely many lines contained in Z that pass through u . By Corollary 3.6 from [16], if Z is singly ruled, then Z contains at most one exceptional point. According to Corollary 2.9, if Z is not ruled then it contains finitely many lines, and thus it cannot contain any exceptional points. Therefore, Z contains at most one exceptional point, and in particular we may assume that z_2, z_3 are not exceptional points. Since z_2 and z_3 are popular but not exceptional, there are infinitely many circles passing through each. On the other hand, at most one circle can pass through the triplet z_1, z_2, z_3 . Thus, after possibly interchanging the roles of z_1, z_2 and z_3 , we may assume that there exists an infinite collection of circles contained in Z that are incident to z_2 but not to z_1 .

Let $\bar{Z} = Z(\bar{g}) \subset \mathbb{R}^3$ be the image of Z after applying the inversion transform around the point z_1 (which we have translated to become the origin) and let $\bar{z}_i = I(z_i)$, $i = 2, 3$. According to Lemma 4.1(b), the infinite family of circles contained in Z that are incident to z_2 but not to z_1 are transformed into an infinite family of circles that are contained in \bar{Z} and incident to \bar{z}_2 . We shall call the latter family $\bar{\mathcal{C}}$.

Consider a plane Π and notice that $\Pi \cap Z(\bar{g})$ is an algebraic curve of degree at most $2E$. This implies that Π contains at most E circles of $\bar{\mathcal{C}}$. Since this holds for any plane, there exists a subset $\bar{\mathcal{C}}' \subset \bar{\mathcal{C}}$ of infinite cardinality such that no two circles in $\bar{\mathcal{C}}'$ are coplanar.

Definition 4.2. If $h \in \mathbb{R}[x_1, x_2, x_3]$ is a polynomial of degree E , we can write $h = \sum_I a_I x^I$, where each index I is of the form (I_1, I_2, I_3) with $I_1 + I_2 + I_3 \leq E$, and $x^I = x_1^{I_1} x_2^{I_2} x_3^{I_3}$. Define

$$h^\dagger = \sum_I a_I x_0^{E-I_1-I_2-I_3} x_1^{I_1} x_2^{I_2} x_3^{I_3}.$$

Then h^\dagger is a homogeneous polynomial of degree E . We define the complex projectivization of the surface $Z(h)$ to be the zero set of h^\dagger in \mathbb{CP}^3 .

Let $\tilde{Z} \subset \mathbb{CP}^3$ be the complex projectivization of the surface \bar{Z} and let

$$\Gamma = \{[x_0 : x_1 : x_2 : x_3] \mid x_0 = 0, x_1^2 + x_2^2 + x_3^2 = 0\}$$

be the *absolute conic* in \mathbb{CP}^3 .

Lemma 4.3. $\tilde{Z} \subset \mathbb{CP}^3$ is irreducible and singly ruled.

Proof. We start by showing that \tilde{Z} is singly ruled. Since $\bar{Z} \subset \mathbb{R}^3$ is ruled, Theorem 2.8 implies $\bar{Z} \subset Z(\text{FL}_{\bar{g}}) \subset \mathbb{R}^3$. This implies that $Z(\bar{g}^\dagger) \subseteq Z(\text{FL}_{\bar{g}}^\dagger)$ holds in \mathbb{CP}^3 . That is, for each point $p \in Z(\bar{g}^\dagger) \subset \mathbb{CP}^3$, there is a line that is contained in $Z(\bar{g}^\dagger)$ and agrees with $Z(\bar{g}^\dagger)$ at p up to order three. According to [20, Theorem 1], this implies that $Z(\bar{g}^\dagger)$, the complex projectivization of $Z(\bar{g})$, is ruled. Moreover, $Z(\bar{g}^\dagger)$ is singly ruled, since if $Z(\bar{g}^\dagger)$ is doubly or triply ruled then \tilde{Z} must be as well, and this is not the case.

It remains is to show that \tilde{Z} is irreducible. It can be easily verified that the homogenization of an irreducible polynomial cannot be reducible. Thus, it suffices to show that the (affine) complexification of $Z(\bar{g})$ is irreducible. This is proved in [30, Lemma 7]. \square

The following arguments are based on the work in [24]. Let

$$\Lambda = \{[x_0 : \dots : x_5] : x_0 x_3 + x_1 x_4 + x_2 x_5 = 0\} \subset \mathbb{CP}^5 \quad (10)$$

be the *Plücker quadric*. There is a bijection between points $p \in \Lambda$ and lines $\ell_p \subset \mathbb{CP}^3$, as given by the *Plücker map*, which sends the point $[x_0 : \dots : x_5] \in \Lambda$ to the line in \mathbb{CP}^3 passing through the points $[x_0 : 0 : -x_5 : x_4]$ and $[0 : x_0 : x_1 : x_2]$. Further details about the Plücker map and Plücker quadric can be found in [11, Section 8.6]. Let $\Lambda_{\tilde{Z}} = \{p \in \Lambda : \ell_p \subset \tilde{Z}\}$; that is, $\Lambda_{\tilde{Z}}$ is the set of all points in Λ that correspond to lines that are fully contained in \tilde{Z} . We claim that $\Lambda_{\tilde{Z}}$ is an algebraic curve in \mathbb{CP}^5 that is composed of a single one-dimensional irreducible component, possibly together with an additional finite set of points. Indeed, $\Lambda_{\tilde{Z}}$ is an algebraic set since the condition $\ell_p \subset \tilde{Z}$ can be written as a condition involving the vanishing of finitely many derivatives of \bar{g}^\dagger (the number will depend on the degree of \bar{g}^\dagger). The only three-dimensional surface that contains a two-dimensional family of lines is a plane. Since \tilde{Z} is ruled but not a plane, $\Lambda_{\tilde{Z}}$ is a one-dimensional set. Finally, to see why $\Lambda_{\tilde{Z}}$ contains a single one-dimensional component, assume, for contradiction, that there exist at least two irreducible one-dimensional components $\Lambda_{\tilde{Z}}^{(1)}, \Lambda_{\tilde{Z}}^{(2)} \subset \Lambda_{\tilde{Z}}$. Since $\bigcup_{p \in \Lambda_{\tilde{Z}}^{(1)}} \ell_p$ is a two-dimensional algebraic variety contained in \tilde{Z} , and since \tilde{Z} is irreducible, then $\bigcup_{p \in \Lambda_{\tilde{Z}}^{(1)}} \ell_p = \tilde{Z}$. A symmetric argument implies $\bigcup_{p \in \Lambda_{\tilde{Z}}^{(2)}} \ell_p = \tilde{Z}$. This contradicts the assumption that \tilde{Z} is singly ruled, since it implies that every point of \tilde{Z} has at least two lines passing through it.

Consider a line ℓ in \tilde{Z} whose image under the Plücker map is the point $[x_0 : \dots : x_5] \in \Lambda$ (here we are using the representation (10) for the Plücker quadric). Then ℓ crosses Γ if and only if $x_0^2 + x_1^2 + x_2^2 = 0$ (recall that ℓ contains the point $[0 : x_0 : x_1 : x_2]$). Thus, the set $\Gamma_\Lambda = \{p \in \Lambda : \ell_p \cap \Gamma \neq \emptyset\}$ is an algebraic variety of codimension 1 in Λ . This in turn implies that either the irreducible one-dimensional component of $\Lambda_{\tilde{Z}}$ is fully contained in Γ_Λ or the intersection $\Lambda_{\tilde{Z}} \cap \Gamma_\Lambda$ is finite. If the former occurs, then all but finitely many lines in \tilde{Z} intersects Γ . However, since \tilde{Z} is the complex projectivization of a real ruled surface, \tilde{Z} contains infinitely many real lines (lines whose defining equation involve only real coefficients), and if ℓ is a real line then $\ell \cap \{x_0 = 0\}$ is a real point. This is a contradiction since the curve Γ contains no real points. Therefore, the intersection $\Lambda_{\tilde{Z}} \cap \Gamma_\Lambda$ is finite.

We will now show that $\Gamma \cap \tilde{Z}$ is a finite set. If this were not the case, then there would be infinitely many points of Γ that are incident to a line of \tilde{Z} . Since every line meets Γ in at most two points (it is easy to show that Γ does not contain any lines), Γ would have intersected infinitely many lines contained in \tilde{Z} . This is a contradiction since according to the previous paragraph, $\Lambda_{\tilde{Z}} \cap \Gamma_\Lambda$ is a finite intersection.

Let $\tilde{\mathcal{C}}'$ be the collection of circles described above, and let $\tilde{\mathcal{C}}'$ be the complex projectivization of these circles. According to the previous paragraph, all of the intersection points between circles of $\tilde{\mathcal{C}}'$ and Γ must lie in the finite intersection $\Gamma \cap \tilde{Z}$. One can show that each circle in $\tilde{\mathcal{C}}'$ is the complex projectivization of a real circle, and every circle of this type intersects Γ in precisely two points. Since $\tilde{\mathcal{C}}'$ contains infinitely many circles, the pigeonhole principle implies that there must exist two circles C_1, C_2 in $\tilde{\mathcal{C}}'$ such that the sets $C_1 \cap \Gamma$ and $C_2 \cap \Gamma$ are identical (each is a set of two points). Recall that, by construction, C_1 and C_2 are contained in two distinct planes Π_1 and Π_2 . Consider the line $\ell = \Pi_1 \cap \Pi_2$ and notice that it contains $C_1 \cap C_2$. Thus, ℓ contains the two intersection points of C_1, C_2 with Γ . Since these two points are contained in the plane $\{x_0 = 0\}$, ℓ is also contained in this plane. However, this is impossible, since ℓ also contains \tilde{z}_2 , which is not in the plane $\{x_0 = 0\}$. This contradiction completes the proof. \square

5 Further applications

High multiplicity points. The following is an easy but interesting consequence of Theorem 1.1.

Corollary 5.1. *Let \mathcal{C} be a set of n circles in \mathbb{R}^3 , and let $q < n$ be an integer so that no sphere or plane contains more than q circles of \mathcal{C} . Let $k \geq k_0$, where k_0 is a sufficiently large constant. Then the number of points incident to at least k circles of \mathcal{C} is*

$$O^* \left(\frac{n^{3/2}}{k^{7/4}} + \frac{n^{3/2}q^{1/2}}{k^3} + \frac{n^{3/2}q^{3/10}}{k^{11/5}} + \frac{n}{k} \right).$$

In particular, if $q = O(1)$, the number of such points is

$$O^* \left(\frac{n^{3/2}}{k^{7/4}} + \frac{n}{k} \right).$$

Proof. Denote the number of such points by m , and observe that they determine at least mk incidences with the circles of \mathcal{C} . Comparing this lower bound with the upper bound in Theorem 1.1, the claim follows. \square

Similar triangles. Another application of Theorem 1.1 is an improved bound on the number of triangles spanned by a set \mathcal{P} of t points in \mathbb{R}^3 and similar to a given triangle Δ . Let $F(t)$ be the maximum number of triangles spanned by the set \mathcal{P} that are similar to the triangle Δ , where the maximum is taken over all triangles Δ and all sets \mathcal{P} with cardinality t (a precise definition can be found in the introduction).

Theorem 5.2.

$$F(t) = O^*(t^{15/7}) = O(t^{2.143}). \quad (11)$$

Proof. Let \mathcal{P} be a set of t points in \mathbb{R}^3 and let $\Delta = uvw$ be a given triangle. Suppose that pqr is a similar copy of Δ , where $p, q, r \in \mathcal{P}$. If p corresponds to u and q to v , then r has to lie on a circle c_{pq} that is orthogonal to the segment pq and whose center lies at a fixed point on this segment. Thus, the number of possible candidates for the point w is exactly the number of incidences between \mathcal{P} and c_{pq} . There are $\binom{t}{2}$ such circles, and no circle arises more than twice in this manner. It follows that $F(t)$ is bounded by the number of incidences between t points and $\binom{t}{2}$ circles. We now apply Theorem 1.1 with $m = t$ and $n = \binom{t}{2}$. It remains to show that the expression (2) is no bigger than $t^{15/7}$.

The first term of The first term of (2) is $O^*(t^{15/7})$. To control the remaining terms, it suffices to show that at most $O\left(\left(\frac{n^3}{m^2}\right)^{3/7}\right) = O(t^{12/7})$ of the circles lie on a common plane or sphere. In fact, we claim that at most $O(t)$ circles can lie on a common plane or sphere. Indeed, let Π be a plane. Then for any circle C_{pq} contained in Π , pq must be orthogonal to Π , and p and q must lie at a fixed distance from Π (the distance is determined by the triangle Δ). This implies that each point of \mathcal{P} can generate at most one circle on Π . The argument for cosphericity is essentially the same. \square

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